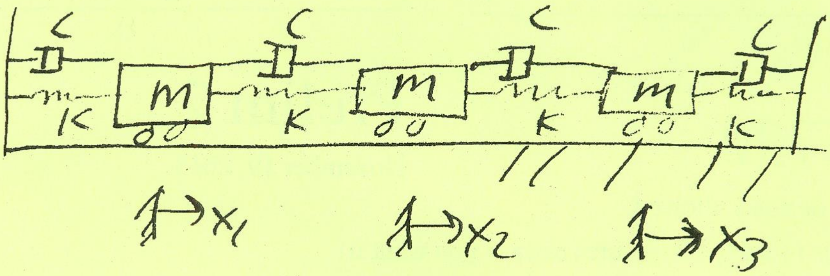
4) Three equal masses m are held apart between two walls by four equal linear springs k and four equal linear dashpots c.



a) Assume $m = 1$, $k = 2$ and $c = 0.01$ in some consistent unit system. Use initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \dot{\vec{x}}(0) = \vec{v}(0) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Write Matlab commands that would find $x_1(t = 10)$. You may *not* use numerical integration (e.g., *no* Euler's method, *no* ODE45, etc).

b) Can the general motion (that is, the solution for an arbitrary initial condition) of this system be found as the sum of solutions each of which is a 'mode' behaving as an independent damped oscillator? If so, describe precisely how to find these modes. If not, explain why not.

c) Assume a force $F = 3\sin(2t)$ is applied to just mass 1. In steady state, approximately what are the amplitudes of vibration of the three masses? No detailed arithmetic is desired, rather say something like 'mass 7 moves much more than mass 8 and much less than mass 4', with appropriate substitutions for 7, 8, 4 the words 'much more' and 'much less'. Use words to justify your answer.

a) FBDs:

$c\dot{x}_1 \leftarrow$ [m] $\rightarrow c(\dot{x}_2-\dot{x}_1)$
$kx_1 \leftarrow$ [m] $\rightarrow k(x_2-x_1)$
$\rightarrow x_1$

$c(\dot{x}_2-\dot{x}_1) \leftarrow$ [m] $\rightarrow c(\dot{x}_3-\dot{x}_2)$
$k(x_2-x_1) \leftarrow$ [m] $\rightarrow k(x_3-x_2)$
$\rightarrow x_2$

$c(\dot{x}_3-\dot{x}_2) \leftarrow$ [m] $\rightarrow -c\dot{x}_3$
$k(x_3-x_2) \leftarrow$ [m] $\rightarrow -kx_3$
$\rightarrow x_3$

LMB:
$$m\ddot{x}_1 = -2kx_1 + kx_2 - 2c\dot{x}_1 + c\dot{x}_2$$
$$m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 + c\dot{x}_1 - 2c\dot{x}_2 + c\dot{x}_3$$
$$m\ddot{x}_3 = kx_2 - 2kx_3 + c\dot{x}_2 - 2c\dot{x}_3$$

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{M}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2c & -c & 0 \\ -c & 2c & -c \\ 0 & -c & 2c \end{bmatrix}}_{C}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}}_{K}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0} \Rightarrow \ddot{\vec{x}} = -M^{-1}(K\vec{x} + C\dot{\vec{x}})$$

First order:
$$\dot{\vec{x}} = \vec{v}$$
$$\dot{\vec{v}} = -M^{-1}K\vec{x} - M^{-1}C\vec{v}$$
$$\Rightarrow \vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix} \quad \dot{\vec{z}} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{A}\begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$$

$$\dot{\vec{z}} = A\vec{z}$$

Solution: $\vec{z} = e^{At}\vec{z}_0$ matrix exponential

```
x0=[1;2;3]; v0=[4;5;6]; z0=[x0;v0];
t=10;
m=1; k=2; c=0.01;
M=[ m  0  0
    0  m  0
    0  0  m ];
C = [ 2*c  -c   0
      -c   2*c  -c
      0    -c   2*c ];
K=[ 2*k  -k   0
    -k   2*k  -k
    0    -k   2k ];
A=[zeros(3,3), eye(3); -inv(M)*K, -inv(M)*C];
Z= expm(A*t)*z0;
x1 = z(1)
```

↑

outputs $x_1(t=10)$

b) Yes, the general solution can be found as the sum of normal modes, since the dashpots are parallel to each spring

$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$

$\vec{x} = M^{-\frac{1}{2}}\vec{q} \Rightarrow MM^{-\frac{1}{2}}\ddot{\vec{q}} + CM^{-\frac{1}{2}}\dot{\vec{q}} + KM^{-\frac{1}{2}}\vec{q} = \vec{0}$

$\ddot{\vec{q}} + M^{-\frac{1}{2}}CM^{-\frac{1}{2}}\dot{\vec{q}} + \underbrace{M^{-\frac{1}{2}}KM^{-\frac{1}{2}}}_{\tilde{K}}\vec{q} = \vec{0}$ $\quad [P, \Lambda] = \text{eig}(\tilde{K})$

$\vec{q} = P\vec{r}$

$\ddot{\vec{r}} + \underbrace{P^T M^{-\frac{1}{2}}CM^{-\frac{1}{2}}}_{\tilde{C}}\dot{\vec{r}} + \Lambda\vec{r} = \vec{0}$

$\ddot{\vec{r}} + \tilde{C}\dot{\vec{r}} + \Lambda\vec{r} = \vec{0}$

Ass~~um~~e $C = \beta K$

$\Rightarrow \ddot{\vec{r}} + \beta\Lambda\dot{\vec{r}} + \Lambda\vec{r} = \vec{0}$

$\ddot{r}_i + \beta\lambda_i\dot{r}_i + \lambda_i r_i = 0$ decoupled (λ_i are diagonal elements of Λ)

$\ddot{r}_i + 2\zeta w_n \dot{r}_i + w_n^2 r_i = 0$

Assume solution: $\vec{x}(t) = \bar{x}e^{-\zeta w_n t}(A\cos(\sqrt{1-\zeta^2}\,t) + B\sin(\sqrt{1-\zeta^2}\,t))$

mode shape: $\bar{x}$, use initial conditions to solve for $\bar{x}, A, B$
(x0, v0)

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4) c) $w = 2$

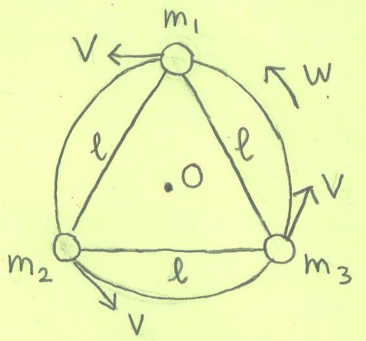
This is the same as the normal mode frequency $w_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2(2)}{1}} = 2$

The mode shape is $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

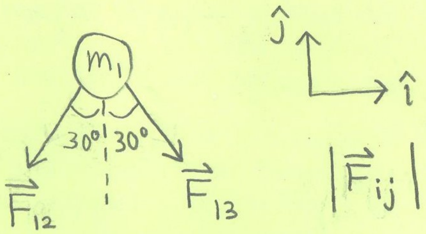
This means:

- mass 2 moves very little compared to the other two masses
- mass 1 and mass 3 move with the same amplitude, but in opposite directions

5) Three equal point masses m move in 2D and are attracted to each other by inverse square gravity $F_{ij} = Gm_i m_j / r_{ij}^2$. There are no other forces. One possible motion is that they all travel in circles about the origin, each with the same constant $\dot{\theta} = \omega$, with the three masses on the vertices of a (rotating) equilateral triangle with sides = ℓ. Find the rate of rotation ω in terms of G, m and ℓ.



FBD:



$$|\vec{F}_{ij}| = \frac{Gm_i m_j}{r_{ij}^2} = \frac{Gm^2}{\ell^2}$$

$$|\vec{F}_{12}| = |\vec{F}_{13}| = \frac{Gm^2}{\ell^2}$$

LMB: $\{m\vec{a} = \vec{F}_{12} + \vec{F}_{13}\}$

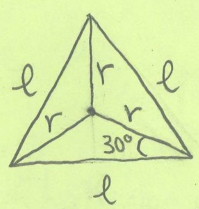
$$\{\ \} \cdot -\hat{j} \Rightarrow ma = |\vec{F}_{12}|\cos 30^\circ + |\vec{F}_{13}|\cos 30^\circ$$

$$ma = \frac{Gm^2}{\ell^2}\frac{\sqrt{3}}{2} + \frac{Gm^2}{\ell^2}\frac{\sqrt{3}}{2}$$

$$a = \frac{\sqrt{3}\,Gm}{\ell^2}$$

Centripetal acceleration:

$$a = \frac{V^2}{r} = \frac{r^2 w^2}{r} = r w^2$$



$$r\cos 30^\circ = \frac{\ell}{2}$$

$$r = \frac{\ell}{\sqrt{3}}$$

$$r w^2 = \frac{\sqrt{3}\,Gm}{\ell^2}$$

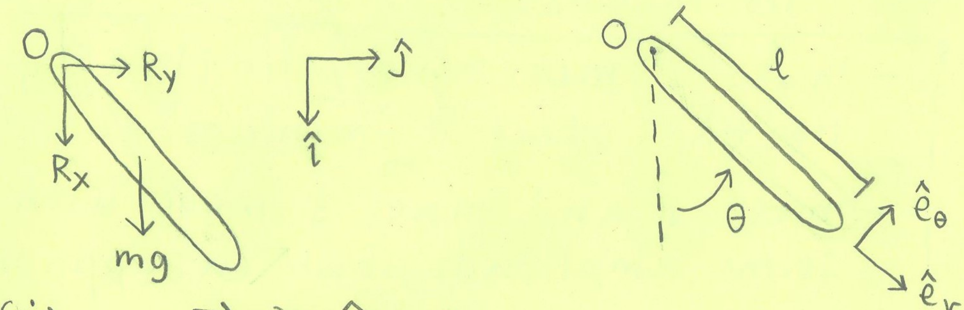
$$\frac{\ell}{\sqrt{3}} w^2 = \frac{\sqrt{3}\,Gm}{\ell^2} \Rightarrow \boxed{w = \sqrt{\frac{3Gm}{\ell^3}}}$$

6) A uniform stick with mass m and length ℓ hangs a stationary hinge at one end. Gravity g acts. The angle of the stick from vertically down is $\theta(t)$, measured CCW.

a) Find $\ddot{\theta}$ in terms of some or all of m, g, ℓ, θ and $\dot{\theta}$ as many different ways as you can. If you have well-defined equations from which $\ddot{\theta}$ could be found, you need not do the algebra.

b) Find, using any single method of your choice, the force acting on the hinge in terms of some or all of $m, g, \ell, \theta, \dot{\theta}$ and any unit vectors you clearly define (your choice of unit vectors).

a) FBD:



$$\text{AMB}/O: \left\{\dot{\vec{H}}_{/O} = \Sigma \vec{M}_{/O}\right\} \cdot \hat{k}$$

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$$I_o \ddot{\theta} = -\frac{\ell}{2} mg \sin\theta \qquad I_o = \frac{1}{3} m\ell^2$$

$$\frac{1}{3}\cancel{m}\ell^2 \ddot{\theta} = \frac{-\ell}{2}\cancel{m}g\sin\theta$$

$$\boxed{\ddot{\theta} = \frac{-3}{2}\frac{g}{\ell}\sin\theta}$$

Lagrange Eqn:

$$\mathcal{L} = E_K - E_P = \frac{1}{2} I_o \dot{\theta}^2 - mg\left(\frac{\ell}{2} - x\right)$$

$$= \frac{1}{6} m\ell^2\dot{\theta}^2 - mg\left(\frac{\ell}{2} - \frac{\ell}{2}\cos\theta\right)$$

$$= \frac{1}{6} m\ell^2\dot{\theta}^2 + \frac{1}{2} mg\ell\cos\theta - \frac{1}{2} mg\ell$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

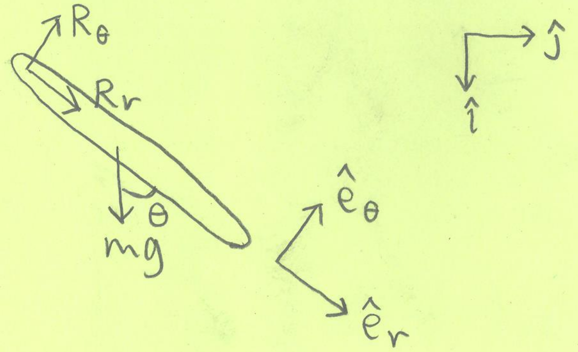
$$-\frac{1}{2} mg\ell\sin\theta - \frac{d}{dt}\left(\frac{1}{3} m\ell^2\dot{\theta}\right) = 0$$

$$-\frac{1}{2}\cancel{m}g\ell\sin\theta - \frac{1}{3}\cancel{m}\ell^2\ddot{\theta} = 0$$

$$\frac{1}{3}\ell^2\ddot{\theta} = -\frac{1}{2} g\ell\sin\theta$$

$$\boxed{\ddot{\theta} = -\frac{3}{2}\frac{g}{\ell}\sin\theta}$$

b)



LMB:

$m\vec{a} = R_r\,\hat{e}_r + R_\theta\,\hat{e}_\theta + mg\,\hat{\imath}$ $\qquad \vec{a} = (\cancel{\ddot{r}} + r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}})\hat{e}_\theta$

$$\{m(r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta) = R_r\hat{e}_r + R_\theta\hat{e}_\theta + mg\hat{\imath}\}$$

$\{\ \}\cdot\hat{e}_r$

$$m r\dot{\theta}^2 = R_r + mg\cos\theta$$

$$\boxed{R_r = mr\dot{\theta}^2 - mg\cos\theta}$$

$\{\ \}\cdot\hat{e}_\theta$

$$mr\ddot{\theta} = R_\theta - mg\sin\theta$$

$$\boxed{R_\theta = mr\ddot{\theta} + mg\sin\theta}$$

Put in terms of $m, g, l, \theta, \dot{\theta}$

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$$r = \frac{l}{2}, \quad \ddot{\theta} = -\frac{3}{2}\frac{g}{l}\sin\theta$$

$$\Rightarrow \boxed{R_r = \frac{ml}{2}\dot{\theta}^2 - mg\cos\theta}$$

$$R_\theta = -\frac{3}{4}m\cancel{l}\frac{g}{\cancel{l}}\sin\theta + mg\sin\theta$$

$$\boxed{R_\theta = \frac{1}{4}mg\sin\theta}$$